Lecture 26

In this lecture we continue our study of counting methods by looking at combinations and permutations.

Permutations & Combinations

$$n! = \begin{cases} n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

$n!$ = the no. of ways to arrange a set of n elements.

E.g. $S = \{a, b, c\}$

a, b, c
a, c, b
b, a, c
b, c, a

c, a, b
c, b, a

$|S| = 3 \qquad 3! = 6$

Arrangements are also called <u>permutations</u>.

Sometimes we want to take k elements from a set of n elements, and arrange them.
How many ways can this be done?

Let's call this $P(n, r)$.

It's sometimes denoted by ${}^{n}P_{r}$.

<u>E.g.</u> $S = \{a, b, c, d\}$

$n = |S| = 4$

Suppose $k = 3$.

We want to take 3 elements from the 4-element set, and arrange them.

This can be done in ..... how many ways?

If we choose $\{a, b, c\}$ as the subset then we know there are 6 possible arrangements.

How many 3-element subsets are there?

$\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$

6 arrangents 6 6 6

So we get 24.

This is also: $4!$

So what does this mean?

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4!$$

<u>Eg.</u>

A club has 7 members. It wants to elect an executive of 3 members, of whom one will be president, another will be secretary and a third will be treasurer.

In how many ways can this be done?

Here we've choosing an executive (a subset) of 3 out of 7, and arranging those 3 in order. So we get

$$P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdots 1}}{\cancel{4 \cdots 1}}$$

$$= 210$$

Now suppose we're just interested in how many 3-person executives can be chosen. We don't care who takes what role.

This brings us to <u>combinations.</u>

The number of combinations of r elements that are possible from an n-element set is

$$\left.\begin{aligned} {}^{n}C_{r} &= C(n,r) \\ &= \binom{n}{r} \end{aligned}\right\} \text{ alternative notations}$$

where

$$\boxed{C(n,r) = \frac{n!}{(n-r)!\, r!}}\,.$$

Then

$$C(n,r) = \frac{n(n-1)\cdots\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3\cdot2\cdot1}}{\left[\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3\cdot2\cdot1}\right]\; r!}$$

$$= \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 3\cdot2\cdot1} \begin{array}{l} \leftarrow r \text{ factors} \\ \leftarrow r \text{ factors} \end{array}$$

E.g. No. of 3-person executives than can be selected from a 7-person club is

$${}_7C_3 = \frac{7\cdot\cancel{6}\cdot5}{\cancel{3\cdot2}\cdot1}$$

$$= 35$$

Note that: $\boxed{P(n,r) = C(n,r) \cdot r!}$

Permutations are "arrangements".

Combinations are "selections".

How many 4-person executives can we select from the 7-person club?

Answer: $${}^{7}C_{4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$
$$= 35$$

Note:

A 4-person executive means a 3-person non-executive group.

So (the no. of ways of selecting a 4-person executive) = (the no. of ways of selecting a 3-person non-executive group)

$= {}^{7}C_{3}$.

Similarly, ${}^{7}C_{5} = {}^{7}C_{2}$.

In general, $\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$.

E.g.

The Tootgarook First Eleven Cricket training squad has 12 members, but only 11 can be selected for the big game against Rosebud.

In how many ways can the team of 11 be selected?

Answer

$$^{12}C_{11} = {}^{12}C_{1} = \frac{12}{1} = 12.$$

Theorem

$$^{n}C_{k} = {}^{n-1}C_{k} + {}^{n-1}C_{k-1}$$

Recall: $\boxed{k \cdot (k-1)! = k!}$

Proof

$$\text{RHS} = {}^{n-1}C_{k} + {}^{n-1}C_{k-1}$$

$$= \frac{(n-1)!}{(n-1-k)!\,k!} + \frac{(n-1)!}{[n-1-(k-1)]!\,(k-1)!}$$

$$= \frac{(n-1)!}{(n-1-k)!\,k!} \cdot \frac{(n-k)}{(n-k)} + \frac{(n-1)!}{(n-k)!\,(k-1)!} \cdot \frac{k}{k}$$

$$= \frac{(n-1)!\,(n-k) + (n-1)!\,k}{(n-k)!\,k!}$$

$$= \frac{(n-1)!\,[(n-k)+k]}{(n-k)!\,k!}$$

$$= \frac{(n-1)!\,n}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,k!}$$

$$= {}^{n}C_{k} = LHS$$